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by

Bruno O. Šubert

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ABSTRACT

A repetitive play of a game against Nature is considered under the assumption that the player knows nothing about the game except his own set of strategies. After each play, he is told the value of the random loss incurred by him. A strategic rule for the player is defined with the property that the average loss achieves asymptotically the minimum functional of the game in probability uniformly in all sequences of Nature's strategies. The rate of convergence of expected average losses is shown as well.

CONTENTS

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Chapter I

INTRODUCTION

Let us consider a sequence of plays of a two-person game, the generic game, where the first player is "Nature." The term "game against Nature" is widely used in both game theory and statistics, but in various senses. Nature is sometimes considered as a player with motivations unspecified or unknown to the other player or sometimes as a player who chooses his strategies so that they form a sequence of independent, identically distributed random variables. Here, we will define Nature as a player who selects his strategies arbitrarily but with no regard whatsoever to the actions of the other player or to the resulting sequence of payoffs. Besides the fact that this notion seems to correspond better to the intuitive idea of Nature as a passive player, our definition is motivated mainly by the application of the problem of repetitive play of a two-person game--called henceforth a sequential game--to statistics. Our concept of Nature implies that she selects the whole sequence of her strategies arbitrarily but once and for all at the beginning of the sequence of plays, which is exactly the case considered in the so-called sequential compound decision problem of mathematical statistics. This concept, which includes the third concept mentioned above (called the empirical Bayes approach in statistics), differs, however, from the first two, which are, in any case, rather vague.

As for the second player--to be referred to as the player--he is considered a player in the true sense of game theory; that is, he is supposed to select his strategies with the intention of minimizing the payoff to his opponents, i.e., his loss. Since we are going to consider the whole situation from the point of view of the player, we will talk about losses rather than payoffs. In the sequential game, it is assumed that the player will utilize for his strategy choices any information about the development of the sequence of plays he may have obtained or inferred during the past plays in the sequence. Of course, he is not supposed to know the sequence of Nature's strategies beforehand; otherwise, his task would be trivial.*

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*Provided he knows the loss function. If not, then this case is included in the case we are considering in this paper.

Since we are dealing with the sequential game, the natural criterion of player's performance in the long run is the average loss incurred by him. It has been indicated (see, e.g., [6], [4], or [9]) that the goal he should try to achieve is reduction of his average loss to the minimum loss of a single game of identical structure, in which Nature (the other player) would use the mixed strategy equal to the empirical distribution up to the point of the pure strategy sequence she is using in the sequential game. The problem thus consists essentially in finding a rule for the player, which would guarantee him that he will achieve this goal, at least asymptotically, no matter what sequence of strategies Nature uses.

In the past decade or two, several papers have dealt with this problem, especially its application to statistics. After the pioneering works of H. Robbins [6] and A. Špaček [8], who both confined themselves to the empirical Bayes problem, two basic strategic rules with the desired property were stated by D. Blackwell in [2], [3], and by J. Hannan in [4]. All the other rules suggested later were derived essentially from one of these two basic rules.

Various approaches to the problem may be classified according to the assumptions made about the information available to the player. This, in turn, may be divided into assumptions about

(1) the knowledge of the generic data of the game, i.e., the sets of strategies and the loss function, and

(2) the data received during the sequence of plays, i.e., for example, the strategies used by Nature in past plays of the sequence or the losses incurred.

Both D. Blackwell and J. Hannan assumed that (1) the player has complete knowledge of the generic game and that (2) after each play, he learns the strategy Nature used. In the statistical version of the sequential game, the variety of possible assumptions in either category 1 and 2 is, of course, much wider. Nevertheless, as far as is known to the author, it has always been assumed that at least (1) the player knows the loss function of the generic game and (2) a random estimate of some sort is available to estimate the empirical distribution of Nature's strategies.

In this paper, we are going to make entirely different assumptions about the player's information. We will assume that:

(1) Except for his own set of strategies from which to choose, the player knows nothing about the generic game; i.e., he knows neither the set of Nature's strategies (which may be infinite) nor the loss function. Moreover, the loss function itself is supposed to be random.

(2) After each play, the player is told the value of the random loss incurred by him in the play.

We will define a rule for the player based on these two requirements. Later we will show that, under relatively moderate assumptions--namely, the set of player's strategies are finite and the random loss function has a nonnegative mean and uniformly bounded third moment--the rule possesses an optimality property similar to those of Blackwell's and Hannan's rules. More precisely, we will show that the difference of the average loss and the appropriate value of the minimum functional of the generic game goes to zero in probability uniformly in all sequences of Nature's strategies.

To illustrate the extent to which the player's information is restricted, let us consider the following simple example. Suppose that both Nature and the player each has two strategies, say $\vartheta^{(0)}$, $\vartheta^{(1)}$ and $a^{(0)}$, $a^{(1)}$, respectively. The player is asked to play repeatedly one of the two games:

GAME 1 (PLAYER across, NATURE down)

	$a^{(0)}$	$a^{(1)}$
$\vartheta^{(0)}$	0	X
$\vartheta^{(1)}$	X	0

GAME 2 (PLAYER across, NATURE down)

	$a^{(0)}$	$a^{(1)}$
$\vartheta^{(0)}$	X	0
$\vartheta^{(1)}$	X	0

where the entry 0 means that the player's loss is zero, while, if the entry is X, a coin is tossed and the player incurs a loss, say $1, if the outcome of the toss is a head, and zero if it is a tail. Clearly, the best rule for game 2 would be to use the strategy $a^{(1)}$ all the time.

On the other hand, the best rule for game 1 will depend on the relative frequencies of each $\vartheta^{(0)}$ and $\vartheta^{(1)}$ in the sequence of Nature's strategies and is, therefore, different from the first one. However, by our assumptions, the player knows nothing about the game but the set $\{a^{(0)}, a^{(1)}\}$ and therefore he cannot distinguish between game 1 and game 2. Thus he cannot decide which of the two rules mentioned he should use, even if he were supplied some information about the sequence Nature is going to use. The rule defined in Chapter III of this paper is, however, invariant with respect to the game structure and allows the player to do as well as if he were told both the relative frequencies of $\vartheta^{(0)}$, $\vartheta^{(1)}$ and which of the two games he has to play.

The rule is relatively simple and more or less suggested by intuition. Before each play in the sequence, the player decides whether the play is going to be a test play (aimed to gain information) or an active play (aimed to minimize the loss). These decisions are based on the outcomes of random experiments--independent flips of a coin where the probability of a head (determining a test play) goes to zero. At a test play, a strategy is chosen randomly with equal probabilities. At an active play, the strategy is selected for which the loss accumulated during the past test plays was minimum. In other words, each strategy is tested from time to time, more and more infrequently but still often enough to guarantee the adequacy of the estimate obtained for the player's decisions.

The rule is defined in Chapter III. Chapter II introduces the notation, basic assumptions, and properties of the generic game. In Chapter IV several lemmas are proven; these are needed for the proofs of Chapter V. Chapter V contains the main theorem (Theorem 1), in which the convergence of average losses is established; Theorems 2 and 3, which give the rate of convergence of expected average losses; and Theorems 4 and 5, which deal with the special case when Nature's moves constitute a sequence of independent identically distributed random variables. Discussion of the results and comments on possible generalizations are contained in Chapter VI.

In this paper, we confine ourselves deliberately to the case of sequential games against Nature and do not extend the results to the more general case of the sequential compound decision problems. Our intention is to investigate the game situation in detail and thus to establish a basis for further extension to the statistical decision case. It has been shown by the author [10] that this can be done. It is hoped that this work may stimulate further effort in this direction.

Chapter II

PREREQUISITES

A. Notation

Throughout this paper, the symbol $(\Omega,\mathcal{A},P)$ will always denote the basic probability space, where Ω with generic elements ω is the set of elementary events, $\mathcal{A}$ is a σ-field of subsets of Ω, and P is a probability measure on $(\Omega,\mathcal{A})$. Random variables will be designated by capital letters, with the argument ω omitted unless necessary. Sets from $\mathcal{A}$ defined as sets of those $\omega\in\Omega$ for which a statement $\mathcal{P}$ is true will be denoted by $\{\mathcal{P}\}$. The indicator function of a set M will be denoted by I_M; the complement of a set $M\subset\Omega$, by M^c. The expectation of a random variable X will be denoted by $E\{X\}$; conditional expectation of X given the σ-field $\mathcal{F}\subset\mathcal{A}$ induced by a random variable Y will be denoted either by $E\{X|Y\}$ or by $E\{X|\mathcal{F}\}$ and used in the sense of the definition in [5], p. 341; similarly, for conditional probability. Other symbols and/or definitions will be used in accordance with [5].

As for other mathematical symbols, a real-valued function f will sometimes be written as $f(.)$ to distinguish it from its value $f(x)$ for the argument x. If f is defined on a finite ordered set A, we will also use the symbol $f(.)$ to denote the Euclidean vector with components $f(a)$, $a\in A$. The symbol R^m will stand for m-dimensional Euclidean space; the components of a vector $x\in R^m$ will be denoted by superscripts in parentheses: $x = (x^{(1)},\dots,x^{(m)})$. The inner product of two vectors $x\in R^m$ and $y\in R^m$ will be denoted by $x\cdot y$. If $\{x_n: n = 1,2,\dots\}$ is a numerical sequence, the symbol $(\overline{x_n})$ will denote the arithmetic mean

$$(\overline{x_n}) = n^{-1}\sum_{k=1}^{n} x_k .$$

If $\{y_n > 0: n = 1,2,\dots\}$ is another sequence of real numbers, the symbol $x_n = O(y_n)$ will designate the property

$$\limsup_{n\to+\infty} \frac{x_n}{|y_n|} < +\infty .$$

B. The Generic Game

As mentioned in the Introduction, a sequential game consists of repetitive plays of a generic game. In this section, we will define and investigate some properties of the latter. Since our model of a sequential game should serve primarily as a basis for the sequential compound decision problem, the terminology and symbols introduced below differ slightly from those common in game theory proper.

Let Θ be an abstract set--the set of parameters ϑ; let $A = \{a^{(1)},\ldots, a^{(m)}\}$--the set of strategies--be a finite set of m elements.

Let $\mathcal{W} = \{W(\vartheta,a):\ \vartheta\epsilon\Theta,\ a\epsilon A\}$ be a two-parameter family of integrable random variables such that for every $W\epsilon\mathcal{W}$

$$0 \le E\{W\} < +\infty . \tag{2.1}$$

We will call $\mathcal{W}$ the random loss function. The triplet $(\Theta,A,\mathcal{W})$ will be referred to as a generic game.

In this paper, we will always assume that the generic game $(\Theta,A,\mathcal{W})$ is nondegenerate in the sense that for every $a\epsilon A$ there exists $\vartheta\epsilon\Theta$ such that

$$E\{W(\vartheta,a)\} \neq 0 . \tag{2.2}$$

Clearly, there is hardly any loss of generality in this assumption since we can always augment the set Θ and the family $\mathcal{W}$ so that (2.2) is true.

In subsequent sections, we will be making some further assumptions concerning integrability of the family $\mathcal{W}$. For reference purposes, let us call

Assumption $(\mathcal{W};\ r)$; $r \ge 1$ an integer: There exists a finite constant C_o such that for every $\vartheta\epsilon\Theta$ and $a\epsilon A$

$$\left[E|W(\vartheta,a)|^r\right]^{1/r} \le C_o .$$

Assumption $(\mathcal{W}; \infty)$: There exists a finite constant C_o such that for every $\vartheta\epsilon\Theta$ and $a\epsilon A$

$$|W(\vartheta,a)| \leq C_o \,. \qquad \text{a.s.}$$

Next, we have to introduce mixed strategies. The set A being finite, a mixed strategy α is simply defined as a vector $\alpha = (\alpha^{(1)},\ldots,\alpha^{(m)})$ from the (m-1)-dimensional probability simplex $\mathcal{A}$ in R^m,

$$\mathcal{A} = \left\{\alpha\epsilon R^m\colon\ \alpha^{(i)} \geq 0;\ i = 1,\ldots,m;\ \sum_{i=1}^{m} \alpha^{(i)} = 1\right\}. \tag{2.3}$$

We will need a similar concept defined for the set of parameters Θ. Let $\mathcal{J}$ be the σ-field of all subsets of Θ; let T be the class of all finite signed measures on the measurable space $(\Theta,\mathcal{J})$ defined by the property: for each $\tau\epsilon T$ there exists a finite set $\{\vartheta_1,\ldots,\vartheta_n\}$ such that

$$\tau(\Theta - \{\vartheta_1,\ldots,\vartheta_n\}) = 0 \,.$$

In other words, all the measures $\tau\epsilon T$ are purely atomic with a finite number of atoms. As an analog of the set of mixed strategies $\mathcal{A}$ will serve the subclass T_o of all probability measures in T. If $\tau\epsilon T_o$ is such that $\tau(\{\vartheta\}) = 1$ for some $\vartheta\epsilon\Theta$, we may write simply ϑ instead of τ.

From now on, let us denote

$$w(\vartheta,a) = E\{W(\vartheta,a)\} \,, \tag{2.4}$$

where $W(\vartheta,a)\epsilon\mathcal{W}$, and let for every $\tau\epsilon T$, $\alpha\epsilon\mathcal{A}$,

$$w(\tau,\alpha) = \int_\Theta \sum_{i=1}^{m} w(\vartheta,a^{(i)})\,\alpha^{(i)}\,d\tau(\vartheta) \,. \tag{2.5}$$

Since, by the definition of the class T, the integral in (2.5) is only a finite sum, $w(.,.)$ is a well defined finite function on $T \times \mathcal{A}$. Moreover, it is easily seen that with addition and multiplication by a constant naturally defined on both T and $\mathcal{A}$, $w(.,.)$ is also a bilinear functional on $T \times \mathcal{A}$.

Let $(\Theta, A, \mathcal{W})$ be a generic game with a random loss function; let for every $\tau \in T$,

$$\varphi(\tau) = \min_{a \in A}\{w(\tau, a)\} \ . \tag{2.6}$$

We will call φ the <u>minimum functional</u> of the generic game $(\Theta, A, \mathcal{W})$.

The minimum functional has a simple property. Let $s^* = (s^{*(1)}, \ldots, s^{*(m)})$ be a mapping from R^m into $\mathcal{A}$ defined for every $x = (x^{(1)}, \ldots, x^{(m)}) \in R^m$ by

$$s^{*(i)}(x) = \left\{ \begin{array}{lll} 1 & \text{if} & x^{(i)} < \min\limits_{\substack{j=1,\ldots,m \\ j \neq i}} \{x^{(j)}\} \\ \alpha^{(i)} & \text{if} & x^{(i)} = \min\limits_{j=1,\ldots,m} \{x^{(j)}\} \\ 0 & \text{if} & x^{(i)} > \min\limits_{j=1,\ldots,m} \{x^{(j)}\} \end{array} \right\} , \tag{2.7}$$

$i = 1, \ldots, m$. Then, clearly, for every $\tau \in T$,

$$\varphi(\tau) = w(\tau, s^*(w(\tau, .))) \ . \tag{2.8}$$

We will use this property to prove the following lemma to be needed later.

<u>Lemma 1</u>. Let $\tau \in T$, $x \in R^m$, $\alpha_1 = s^*(w(\tau,.) + x)$, $\alpha_2 \in \mathcal{A}$.

Then

$$w(\tau, \alpha_1) - w(\tau, \alpha_2) \leq x \cdot (\alpha_2 - \alpha_1) \ . \tag{2.9}$$

<u>Proof</u>.

Let $t_x \in T$ be such that $w(t_x, .) = x$. It is easy to see that such t_x exists for every $x \in R^m$ since, by assumption (2.2), we can always find $\{\vartheta_1, \ldots, \vartheta_m\} \subset \Theta$ such that $w(\vartheta_i, a^{(i)}) \neq 0$ for every $i = 1, \ldots, m$ and then set $t_x(\{\vartheta_i\}) = x^{(i)}[w(\vartheta_i,a^{(i)})]^{-1}$; $i = 1,\ldots,m$; and $t_x(\Theta - \{\vartheta_1,\ldots,\vartheta_m\}) = 0$.

Next, by (2.6) and (2.7)

$$\varphi(\tau + t_x) = w(\tau + t_x,\alpha_1) \leq w(\tau + t_x,\alpha_2) \ , \tag{2.10}$$

which implies

$$0 \leq w(\tau + t_x,\alpha_2) - w(\tau + t_x,\alpha_1) \ . \tag{2.11}$$

Finally, linearity of $w(.,a)$ gives

$$w(\tau + t_x,\alpha_\lambda) = w(\tau,\alpha_\lambda) + x\cdot\alpha_\lambda; \quad \lambda = 1,2 \ ; \tag{2.12}$$

which together with (2.11) gives (2.9).

This completes the Proof of Lemma 1.

C. The Sequential Game

In the sequential game against Nature, Nature first selects a sequence of parameters $\vartheta_1,\vartheta_2,\ldots$ from the parameter space Θ. For further purposes, we will assume that the sequence always begins with a "dummy" parameter ϑ_o such that

$$W(\vartheta_o,.) = W_o(\vartheta_o,.) = 0 \ . \tag{2.13}$$

We will denote the set of all sequences $\underline{\vartheta} = \{\vartheta_n: n = 0,1,\ldots\}$ by Θ^∞, this representing the set of all Nature's strategies in the sequential game. With every sequence $\underline{\vartheta} = \{\vartheta_n: n = 0,1,\ldots\}\epsilon\Theta^\infty$, we will associate a sequence $\{\tau_n \epsilon T: n = 0,1,\ldots\}$ defined by

$$\tau_n(B) = \frac{1}{n}\sum_{k=1}^{n} I_B(\vartheta_k); \ B\epsilon\mathcal{T}; \ n = 1,2,\ldots; \ \tau_o(.) = 0 \ . \tag{2.14}$$

Thus, for $n = 1,2,\ldots$, $\tau_n \epsilon T_o$ is the empirical distribution of ϑ defined by $\{\vartheta_1,\ldots,\vartheta_n\}$, and $\tau_n(B)$ is the proportion of ϑ_k's, $k = 1,\ldots,n$, in B.

Let $\{a_n: n = 1,2,\ldots\}$ be a sequence of pure strategies used by the player. The result of the sequential game is then a sequence of random losses defined as a sequence of independent random variables

$$\{W_n(\vartheta_n, a_n): n = 1,2,\ldots\} , \tag{2.15}$$

where the random variable $W_n(\vartheta_n, a_n)$ is distributed as $W(\vartheta, a) \in \mathcal{U}$ for $\vartheta = \vartheta_n$, $a = a_n$.

As mentioned before, the player selects his mixed strategies at each play on the basis of his past experience, using for his choices a strategic rule, which tells him for each $n = 1,2,\ldots$ the mixed strategy S_n he has to use at the n-th play. Since each S_n thus depends on the outcomes of the past plays and since the strategic rule itself may be a randomized rule, the sequence $\{S_n: n = 1,2,\ldots\}$--to be called the sequence of mixed strategies generated by the rule--is a sequence of random vectors with values in $\mathcal{a}$.

A strategic rule also generates a sequence of pure strategies $\{\Psi_n: n = 1,2,\ldots\}$ defined as a sequence of random variables with values in A such that if $\mathcal{F}_n$ denotes the σ-field induced by the family $\{S_1,\ldots, S_n; W_1(\vartheta_1,.),\ldots,W_n(\vartheta_n,.)\}$ then for every $n = 1,2,\ldots$; $\underline{\vartheta} \in \Theta^\infty$, $i = 1,\ldots,m$

$$P\left\{\{\Psi_n = a^{(i)}\} \middle| \mathcal{F}_n\right\} = S_n^{(i)}, \qquad \text{a.s.} \tag{2.16}$$

and

$$\Psi_n \text{ and } W_n(\vartheta_n,.) \text{ are independent.} \tag{2.17}$$

Chapter III

THE STRATEGIC RULE

In this chapter, we will define a strategic rule satisfying the assumptions we made about the information available to the player. Later (Chapter V) we will show that this rule is <u>weakly asymptotically optimal</u> in the sense that if $\{\Psi_n: n = 1,2,\ldots\}$ is the sequence of pure strategies generated by it, then

$$\frac{1}{n}\sum_{k=1}^{n} W_k(\vartheta_k,\Psi_k) - \varphi(\tau_n) \xrightarrow{P} 0$$

uniformly in $\underline{\vartheta}\in\vartheta^\infty$. This means that, by using this rule, the player can do as well as if he knew all the data about the game structure and if he were told the asymptotic empirical distribution of the ϑ's in the sequence Nature is going to use. Moreover, he can do this uniformly in all Nature's possible choices. More precisely, given $\epsilon > 0$ and $\delta > 0$, there exists a positive integer $n(\epsilon,\delta)$ with the property that, if the number of plays exceeds $n(\epsilon,\delta)$, the average loss will differ from the goal φ more than ϵ with probability less than δ no matter what the sequence of ϑ's was used by Nature. The integer $n(\epsilon,\delta)$ can be obtained from Theorem 2 or Theorem 3 of Chapter V, which give the rate of convergence.

Let

$$\{U_n: n = 0,1,\ldots\} \tag{3.1}$$

be a sequence of independent random variables, taking values 0 and 1, and let for every $n = 0,1,\ldots$

$$P\{U_n = 1\} = p_n > 0 \ . \tag{3.2}$$

Further, let

$$\{V_n: n = 0,1,\ldots\} \tag{3.3}$$

be a sequence of independent identically distributed random vectors taking values in the set $\mathcal{Q}$ and such that for every $i = 1,\ldots,m$,

$$P\left\{V_n = \left(\delta_1^i,\ldots,\delta_m^i\right)\right\} = m^{-1} , \tag{3.4}$$

where δ_j^i is the Kronecker delta.

The random variable U_n determines whether the n-th play will be a test play ($U_n = 1$) or an active play ($U_n = 0$), while the V_n determines the strategy to be used in a test play.

The sequences (3.1) and (3.3), as well as the sequence (2.15), are also assumed to be mutually independent (for every $\underline{\vartheta}\epsilon\Theta^{\infty}$).

Next, let

$$\{Y_n: n = 0,1,\ldots\} \tag{3.5}$$

be a sequence of random vectors with values in R^m defined for every $\underline{\vartheta}\epsilon\Theta^{\infty}$; $n = 0,1,\ldots$ by

$$Y_n = U_n m p_n^{-1}\left(\eta + W_n(\vartheta_n,.)\right) \cdot V_n , \tag{3.6}$$

where $\eta = (\eta^{(1)},\ldots,\eta^{(m)})$ is the vector with all components $\eta^{(i)} = 1$. Thus the vector Y_n has either all components zero (if $U_n = 0$) or has only one nonzero component, namely that one for which $V_n^{(i)} = 1$, the component being then equal to $mp_n^{-1}(1 + W_n(\vartheta_n,i))$.

The sequence (3.5) therefore represents the information the player is receiving along the sequence of test plays, namely, the pure strategy used and the loss incurred.

<u>The strategic rule</u> we are suggesting is defined by means of the sequence of mixed strategies $\{S_n: n = 1,2,\ldots\}$ generated by it as follows:

$$S_n = U_n V_n + (1 - U_n) s^*\left(\sum_{k=0}^{n-1} Y_k\right), \tag{*}$$

where s^* is the mapping (2.7).

Notice that

$$S_n = s^*\left(\sum_{k=0}^{n-1} Y_k\right)$$

if the n-th play is an active one, thus selecting the strategy $a^{(i)} \epsilon A$ for which

$$\sum_{k=0}^{n-1} Y_k^{(i)}$$

is minimum and in a test play $S_n = V_n$ selects any $a \epsilon A$ with equal likelihood.

We will refer to this strategic rule as to the strategic rule (*).

Chapter IV

SOME AUXILIARY RESULTS

In this chapter, we are going to prove several lemmas, which will be used in the next chapter.

Let $\underline{\vartheta} = \{\vartheta_n: n = 0,1,\ldots\} \in \Theta^\infty$ be a sequence of ϑ's; let $\{Y_n: n = 0,1,\ldots\}$ be the sequence of random vectors (3.5). By the assumption of integrability of the random loss function

$$E\left\{Y_n^{(i)}\right\} = 1 + w\left(\vartheta_n, a^{(i)}\right); \; i = 1,\ldots,m \tag{4.1}$$

exists and is finite so that we can center the random vectors Y_n at expectations:

$$\tilde{Y}_n = \left(\tilde{Y}_n^{(1)},\ldots,\tilde{Y}_n^{(m)}\right); \; \tilde{Y}_n^{(i)} = Y_n^{(i)} - E\left\{Y_n^{(i)}\right\}; \; i = 1,\ldots,m \,. \tag{4.2}$$

We will need bounds for absolute moments of the random variables $\tilde{Y}_n^{(i)}$. Let us assume that the assumption $(\mathcal{W};r)$ holds for some $r \geq 3$, and let $1 \leq r' \leq r$. We have

$$\begin{aligned} E\left\{\left|\tilde{Y}_n^{(i)}\right|^{r'} \Big| W_n\left(\vartheta_n, a^{(i)}\right)\right\} &= \left|mp_n^{-1}\left[1 + W_n\left(\vartheta_n, a^{(i)}\right)\right] - \left[1 + w\left(\vartheta_n, a^{(i)}\right)\right]\right|^{r'} m^{-1}p_n \\ &\quad + \left|-\left[1 + w\left(\vartheta_n, a^{(i)}\right)\right]\right|^{r'}\left(1 - m^{-1}p_n\right) \\ &\leq 2^{r'-1}m^{r'-1}p_n^{r'-1}\left[\left|1 + W_n\left(\vartheta_n, a^{(i)}\right)\right|^{r'} \right. \\ &\quad \left. + 2\left|1 + w\left(\vartheta_n, a^{(i)}\right)\right|^{r'}\right] \end{aligned} \tag{4.3}$$

by the C_r-inequality ([5], p. 155) and the fact that

$$m^{-1}p_n \leq 1 \leq m^{r'-1}p_n^{1-r'} \,.$$

Further, by $(\mathfrak{W};r)$ there exists a finite constant C_1 such that, say

$$E|1 + W_n(\vartheta,a)|^{r'} \leq C_1^{r'} \text{ for every } \vartheta\epsilon\Theta,\ a\epsilon A\ , \tag{4.4}$$

and consequently by Jensen Inequality also

$$|1 + w(\vartheta,a)|^{r'} \leq C_1^{r'} \text{ for every } \vartheta\epsilon\Theta,\ a\epsilon A\ . \tag{4.5}$$

Hence and from (4.3) we have immediately

$$E|\tilde{Y}_n^{(i)}|^{r'} \leq (2C_1)^{r'}\ m^{r'-1} p_n^{1-r'}\ . \tag{4.6}$$

Next, let us introduce the random variables

$$\tilde{Y}_n^{(ij)} = \tilde{Y}_n^{(i)} - \tilde{Y}_n^{(j)};\ i = 1,\ldots,m;\ j = 1,\ldots,m;\ n = 0,1,\ldots\ . \tag{4.7}$$

By the C_r-inequality and (4.6) we have now

$$|\tilde{Y}_n^{(ij)}|^{r'} \leq (4C_1)^{r'}\ m^{r'-1} p_n^{1-r'}\ . \tag{4.8}$$

We will need also a lower bound for the variance of $\tilde{Y}_n^{(ij)}$, $i \neq j$. Direct computation gives

$$E\left\{|\tilde{Y}_n^{(ij)}|^2 \middle| W_n\left(\vartheta_n, a^{(i)}\right),\ W_n\left(\vartheta_n, a^{(j)}\right)\right\}$$

$$= mp_n^{-1}\left[E\left|1 + W_n\left(\vartheta_n, a^{(i)}\right)\right|^2 + E\left|1 + W_n\left(\vartheta_n, a^{(j)}\right)\right|^2\right] - |\Delta_n^{(ij)}|^2\ , \tag{4.9}$$

where we denoted

$$\Delta_n^{(ij)} = w\left(\vartheta_n, a^{(i)}\right) - w\left(\vartheta_n, a^{(j)}\right)\ . \tag{4.10}$$

Using again Jensen Inequality

$$E|1 + W_n(\vartheta_n,a)|^2 \geq |1 + w(\vartheta_n,a)|^2\ , \tag{4.11}$$

we obtain from (4.9)

$$E|\tilde{Y}_n^{(ij)}|^2 \geq 2mp_n^{-1}\left[1 + w\left(\vartheta_n, a^{(i)}\right) + w\left(\vartheta_n, a^{(j)}\right)\right]$$

$$+ \left(mp_n^{-1} - 1\right)\left[w^2\left(\vartheta_n, a^{(i)}\right) + w^2\left(\vartheta_n, a^{(j)}\right)\right]$$

$$+ 2w\left(\vartheta_n, a^{(i)}\right)w\left(\vartheta_n, a^{(j)}\right) \geq 2mp_n^{-1}$$

$$+ \left(mp_n^{-1} - 1\right)\left[w^2\left(\vartheta_n, a^{(i)}\right) + w^2\left(\vartheta_n, a^{(j)}\right)\right] \tag{4.12}$$

since $w(.,.)$ is nonnegative [assumption (2.1)]. Hence, if $p_n \leq m^{-1}$, we have

$$E|\tilde{Y}_n^{(ij)}|^2 \geq 2mp_n^{-1} . \tag{4.13}$$

Thus we have proven

Lemma 2. Let $\tilde{Y}_n^{(i)}$ and $\tilde{Y}_n^{(ij)}$; $i = 1,\ldots,m$; $j = 1,\ldots,m$; be the random variables (4.2) and (4.7), respectively; let assumption $(\mathcal{W};r)$ for some $r \geq 3$, be satisfied. Then there is a constant $C_1 < +\infty$ such that for every $n = 0,1,\ldots$ and $1 \leq r' \leq r$

$$E|\tilde{Y}_n^{(i)}|^{r'} \leq (2C_1)^{r'} m^{r'-1} p_n^{1-r'} , \tag{4.14}$$

and

$$E|\tilde{Y}_n^{(ij)}|^{r'} \leq (4C_1)^{r'} m^{r'-1} p_n^{1-r'} . \tag{4.15}$$

If, moreover, $p_n \leq m^{-1}$, then for $i \neq j$

$$E|\tilde{Y}_n^{(ij)}|^2 \geq 2mp_n^{-1} . \tag{4.16}$$

Since the random variables $\tilde{Y}_n^{(ij)}$; $n = 0,1,\ldots$; are independent, their consecutive sums have another property expressed by

Lemma 3. Let

$$Z_n^{(ij)} = n^{-1/2} \sum_{k=0}^{n} \tilde{Y}_k^{(ij)} , \tag{4.17}$$

$n = 1,2,\ldots$; $i = 1,\ldots,m$; $j = 1,\ldots,m$; $i \neq j$, let $p_n \leq m^{-1}$; $n = 0,1,\ldots$; let assumption $(\mathfrak{W};3)$ hold. Let $J(x_1,x_2)$ denote either of the intervals $[x_1,x_2)$, $(x_1,x_2]$, where $-\infty < x_1 \leq x_2 < +\infty$. Then

$$P\left\{Z_n^{(ij)} \in J(x_1,x_2)\right\} \leq (x_2 - x_1)(2m)^{-1/2}\, n^{1/2}\left(\sum_{k=0}^{n} p_k^{-1}\right)^{-1/2}$$

$$+ 32C_1^3\beta\, m^{1/2} \sum_{k=0}^{n} p_k^{-2}\left(\sum_{k=0}^{n} p_k^{-1}\right)^{-3/2} , \tag{4.18}$$

where C_1 is the constant from Lemma 2 and β is the Berry-Esseen constant.

Proof.

Let $\hat{F}_n^{(ij)}$ denote the distribution function of the law

$$\mathfrak{L}\left[\left(\sum_{k=0}^{n} E|\tilde{Y}_k^{(ij)}|^2\right)^{-1/2} \sum_{k=0}^{n} \tilde{Y}_k^{(ij)}\right];$$

let G be the distribution function of the normal law $\mathfrak{N}(0,1)$. Since the random variables $\tilde{Y}_n^{(ij)}$ are independent, centered at expectations and, by Lemma 2, have positive variances, the Berry-Esseen normal approximation theorem ([5], p. 288) applies, yielding

$$\sup_{x\in(-\infty,+\infty)} |\hat{F}_n^{(ij)}(x) - G(x)| \leq \beta \sum_{k=0}^{n} E|\tilde{Y}_k^{(ij)}|^3 \left(\sum_{k=0}^{n} E|\tilde{Y}_k^{(ij)}|^2 \right)^{-3/2} . \qquad (4.19)$$

Let $F_n^{(ij)}$ be the distribution function of the law $\mathfrak{L}(Z_n^{(ij)})$; let

$$\bar{\sigma}_n^{(ij)} = \left(\frac{1}{n} \sum_{k=0}^{n} E|\tilde{Y}_k^{(ij)}|^2 \right)^{1/2} .$$

Since

$$\left(\sum_{k=0}^{n} E|\tilde{Y}_k^{(ij)}|^2 \right)^{-1/2} \sum_{k=0}^{n} \tilde{Y}_k^{(ij)} = \left(\bar{\sigma}_n^{(ij)}\right)^{-1} Z_n^{(ij)}$$

we have

$$\hat{F}_n^{(ij)}(x) = F_n^{(ij)}\left(x\bar{\sigma}_n^{(ij)}\right) .$$

Further, using Lemma 2, we find

$$\sum_{k=0}^{n} E|\tilde{Y}_k^{(ij)}|^3 \left(\sum_{k=0}^{n} E|\tilde{Y}_k^{(ij)}|^2 \right)^{-3/2} < 16C_1^3\, m^{1/2} \sum_{k=0}^{n} p_k^{-2} \left(\sum_{k=0}^{n} p_k^{-1} \right)^{-3/2} .$$

Thus (4.19) becomes

$$\sup_{x\in(-\infty,+\infty)} \left| F_n^{(ij)}(x) - G\left[x\left(\bar{\sigma}_n^{(ij)}\right)^{-1}\right] \right| < 16C_1^3\, m^{1/2} \sum_{k=0}^{n} p_k^{-2} \left(\sum_{k=0}^{n} p_k^{-1} \right)^{-3/2} . \qquad (4.20)$$

Finally, by the well known property of the distribution function G and Lemma 2,

$$\left| G\left[x_2\left(\bar{\sigma}_n^{(ij)}\right)^{-1}\right] - G\left[x_1\left(\bar{\sigma}_n^{(ij)}\right)^{-1}\right] \right| \leq (x_2 - x_1)\left(\bar{\sigma}_n^{(ij)}\right)^{-1}$$

$$\leq (x_2 - x_1)(2m)^{-1/2}\, n^{1/2} \left(\sum_{k=0}^{n} p_k^{-1} \right)^{-1/2} . \qquad (4.21)$$

Inequalities (4.20) and (4.21), together with the fact that the Berry-Essen Theorem holds both for right-continuous and left-continuous distribution function $F_n^{(ij)}$ prove (4.18).

The following lemma is a trivial generalization of the law of large numbers.

<u>Lemma 4</u>. Let $\{X_n: n = 1,2,\ldots\}$ be a sequence of random variables with distributions depending on a sequence of parameters $\underline{\vartheta} = \{\vartheta_n: n = 1,2,\ldots\}\epsilon\Theta^{\infty}$; let $\{\mathcal{F}_n: n = 0,1,\ldots\}$ be a sequence of sub-σ-field of $\mathcal{S}$. If

$$n^{-2}\sum_{k=1}^{n} E|X_k|^2 \to 0 \text{ uniformly in } \underline{\vartheta}\epsilon\Theta^{\infty} \tag{4.22}$$

and if the family $\{X_1,\ldots,X_n\}$ is $\mathcal{F}_n$-measurable for every $n = 1,2,\ldots$, $\underline{\vartheta}\epsilon\Theta^{\infty}$, then

$$\frac{1}{n}\sum_{k=1}^{n} (X_k - E\{X_k|\mathcal{F}_{k-1}\}) \xrightarrow{P} 0 \text{ uniformly in } \underline{\vartheta}\epsilon\Theta^{\infty} . \tag{4.23}$$

<u>Proof of Lemma 4</u>.

The proof is straightforward. Let $\epsilon > 0$, let $X_n' = X_n - E\{X_n|\mathcal{F}_{n-1}\}$. By Tchebichev Inequality

$$P\left\{\left|\frac{1}{n}\sum_{k=1}^{n} X_k'\right| \geq \epsilon\right\} \leq \frac{1}{n^2\epsilon^2} E\left|\sum_{k=1}^{n} X_k'\right|^2 . \tag{4.24}$$

However, since by assumption $\{X_1,\ldots,X_n\}$ is $\mathcal{F}_n$-measurable, the random variables X_n' are centered at expectations given the predecessors so that the extended Bienaymé Equality ([5], p. 386) holds. Therefore

$$E\left|\sum_{k=1}^{n} X_k'\right|^2 = \sum_{k=1}^{n} E|X_k'|^2 \leq \sum_{k=1}^{n} E|X_k|^2 ,$$

which, together with (4.24) and assumption (4.22) gives (4.23).

This completes the Proof of Lemma 4.

Later, we will use one simple lemma on sequences of positive real numbers.

Lemma 5. Let $\{p_n: n = 0,1,\ldots\}$ be a sequence of positive numbers such that for every $n = 0,1,\ldots$

$$(n+1)p_{n+1} \geq np_n \ . \tag{4.25}$$

Then

$$\liminf_{n\to+\infty} \frac{1}{np_n^{-1}} \sum_{k=0}^{n-1} p_k^{-1} > 0 \ . \tag{4.26}$$

Proof.

By (4.25) we have for every $n = 1,2,\ldots$

$$p_k^{-1} \geq \frac{k}{n} p_n^{-1}, \ k = 0,\ldots,n-1 \ . \tag{4.27}$$

Hence

$$\frac{1}{np_n^{-1}} \sum_{k=0}^{n-1} p_k^{-1} \geq \frac{1}{n} \sum_{k=0}^{n-1} \frac{k}{n} = \frac{n-1}{2n} \to \frac{1}{2} \tag{4.28}$$

as $n \to +\infty$.

Lemma 5 is proven.

The remaining three lemmas constitute essential parts of the theorems in the next chapter.

Lemma 6. Let $\{Y_n: n = 0,1,\ldots\}$ and $\{\tilde{Y}_n: n = 0,1,\ldots\}$ be the sequences of random vectors (3.5) and (4.2), respectively; let

$$S'_n = s^*\left(\sum_{k=0}^{n} Y_k\right), \ n = 0,1,\ldots \ , \tag{4.29}$$

and let

$$(\mathcal{U};r) \text{ hold for some } r \geq 3 \; . \tag{4.30}$$

If the sequence $\{p_n: n = 0,1,\ldots\}$ of probabilities (3.2) satisfies the conditions

$$p_n \downarrow 0 \; , \tag{4.31}$$

and

$$np_n \uparrow +\infty \; , \tag{4.32}$$

then

$$\frac{1}{n}\sum_{k=1}^{n} \tilde{Y}_k \cdot S_k' \overset{P}{\to} 0 \text{ uniformly in all sequences } \underset{\sim}{\vartheta}\in\Theta^{\infty} \; . \tag{4.33}$$

Proof.

Let $\tilde{Y}_n^{(ij)}$ be the random variables (4.7). Since $S_n' \in \mathcal{Q}$ for all $n = 0,1,\ldots,$ we have the identity

$$\frac{1}{n}\sum_{k=1}^{n} \tilde{Y}_k \cdot S_k' = \frac{1}{m}\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{1}{n}\sum_{k=0}^{n}\tilde{Y}_k^{(ij)}S_k'^{(i)} + \frac{1}{m}\sum_{i=1}^{m}\frac{1}{n}\sum_{k=1}^{n}\tilde{Y}_k^{(i)} \; . \tag{4.34}$$

By Lemma 2 (4.14) and the condition (4.31), we have for every $j = 1,\ldots,m$; $n = 1,2,\ldots$

$$n^{-2}\sum_{k=1}^{n} E|\tilde{Y}_k^{(i)}| \leq (2C_1)^2 mn^{-2}\sum_{k=1}^{n} p_k^{-1} \leq (2C_1)^2 m(np_n)^{-1} \; , \tag{4.35}$$

which goes to zero by (4.32). Since the random variables $\tilde{Y}_n^{(i)}$ are independent and centered at expectations, the weak law of large numbers (or Lemma 4 with $\mathcal{F}_n$ induced by $\{\tilde{Y}_1^{(i)},\ldots,\tilde{Y}_n^{(i)}\}$) yields

$$\frac{1}{n}\sum_{k=1}^{n}\tilde{Y}_k^{(i)} \overset{P}{\to} 0 \text{ uniformly in } \underset{\sim}{\vartheta}\in\Theta^{\infty} \; . \tag{4.36}$$

In view of (4.34), it remains to prove that for any $i = 1,\ldots,m$, $j = 1,\ldots,m$, $i \neq j$,

$$\frac{1}{n}\sum_{k=1}^{n} \tilde{Y}_k^{(ij)} S_k'^{(i)} \xrightarrow{P} 0 \text{ uniformly in } \underline{\vartheta}\in\Theta^{\infty} . \tag{4.37}$$

Let $\mathcal{Y}_n$, $n = 0,1,\ldots,$ be the σ-field induced by the family $\{Y_o,\ldots,Y_n\}$. Since, clearly,

$$\left\{\tilde{Y}_o^{(ij)} S_o'^{(i)},\ldots,\tilde{Y}_n^{(ij)} S_n'^{(i)}\right\}$$

is $\mathcal{Y}_n$-measurable for every $n = 0,1,\ldots,$ and since by Lemma 2 (4.15), (4.31), and (4.32),

$$n^{-2}\sum_{k=1}^{n} E|\tilde{Y}_k^{(ij)} S_k'^{(i)}|^2 \leq n^{-2}\sum_{k=1}^{n} E|\tilde{Y}_k^{(ij)}|^2 \rightarrow 0 \text{ uniformly in } \underline{\vartheta}\in\Theta^{\infty} , \tag{4.38}$$

we obtain (4.37) from Lemma 4 if we prove that for any $i \neq j$

$$\frac{1}{n}\sum_{k=1}^{n} E\left\{\tilde{Y}_k^{(ij)} S_k'^{(i)} \middle| \mathcal{Y}_{k-1}\right\} \xrightarrow{P} 0 \text{ uniformly in } \underline{\vartheta}\in\Theta^{\infty} . \tag{4.39}$$

Let

$$H_n^{(i)} = \left\{\sum_{k=0}^{n} Y_k^{(i)} < \min_{\substack{\ell=1,\ldots,m \\ \ell\neq i}} \sum_{k=0}^{n} Y_k^{(\ell)}\right\} , \tag{4.40}$$

$$K_n^{(i)} = \left\{\sum_{k=0}^{n} Y_k^{(i)} \leq \min_{\ell=1,\ldots,m} \sum_{k=0}^{n} Y_k^{(\ell)}\right\} , \tag{4.41}$$

$$L_n^{(i)} = \left\{\sum_{k=0}^{n} Y_k^{(i)} = \min_{\ell=1,\ldots,m} \sum_{k=0}^{n} Y_k^{(\ell)}\right\} , \tag{4.42}$$

so that by the definition of S'_n

$$S'^{(i)}_n = I_{H_n^{(i)}} + \alpha^{(i)} I_{L_n^{(i)}} ,$$

whence, since $0 \le \alpha^{(i)} \le 1$ and $H_n^{(i)} \cup L_n^{(i)} = K_n^{(i)}$,

$$I_{H_n^{(i)}} \le S'^{(i)}_n \le I_{K_n^{(i)}} . \tag{4.43}$$

Therefore, almost surely

$$\begin{aligned}
&\min\left\{E\left\{\tilde{Y}_n^{(ij)} I_{H_n^{(i)}} \middle| \mathcal{Y}_{n-1}\right\}, E\left\{\tilde{Y}_n^{(ij)} I_{K_n^{(i)}} \middle| \mathcal{Y}_{n-1}\right\}\right\} \\
&\quad \le E\left\{\tilde{Y}_n^{(ij)} S'^{(i)}_n \middle| \mathcal{Y}_{n-1}\right\} \\
&\quad \le \max\left\{E\left\{\tilde{Y}_n^{(ij)} I_{H_n^{(i)}} \middle| \mathcal{Y}_{n-1}\right\}, E\left\{\tilde{Y}_n^{(ij)} I_{K_n^{(i)}} \middle| \mathcal{Y}_{n-1}\right\}\right\} .
\end{aligned} \tag{4.44}$$

We will consider first

$$E\left\{\tilde{Y}_n^{(ij)} I_{H_n^{(i)}} \middle| \mathcal{Y}_{n-1}\right\} . \tag{4.45}$$

Let $\{\gamma_n : n = 1,2,\ldots\}$ be a sequence of positive real numbers (truncating constants); let

$$\Gamma_n^{(i\ell)} = \left\{|\tilde{Y}_n^{(i\ell)}| \le \gamma_n\right\}; \ \ell = 1,\ldots,m , \tag{4.46}$$

$$\Gamma_n^{(i)} = \bigcap_{\substack{\ell=1 \\ \ell \ne i}}^{m} \Gamma_n^{(i\ell)} . \tag{4.47}$$

Since clearly

$$\left|\tilde{Y}_n^{(ij)} I_{H_n^{(i)}} - \tilde{Y}_n^{(ij)} I_{H_n^{(i)} \cap \Gamma_n^{(i)}}\right| \le \left|\tilde{Y}_n^{(ij)}\right| I_{\left(\Gamma_n^{(i)}\right)^c} , \tag{4.48}$$

and since the random variable $|\tilde{Y}_n^{(ij)}| I_{\left(\Gamma_n^{(i)}\right)^c}$ is independent of $\mathcal{Y}_{n-1}$ we have almost surely

$$\left|E\left\{\tilde{Y}_n^{(ij)} I_{H_n^{(i)}} \middle| \mathcal{Y}_{n-1}\right\} - E\left\{\tilde{Y}_n^{(ij)} I_{H_n^{(i)} \cap \Gamma_n^{(i)}} \middle| \mathcal{Y}_{n-1}\right\}\right| \le E\left\{|\tilde{Y}_n^{(ij)}| I_{\left(\Gamma_n^{(i)}\right)^c}\right\} \tag{4.49}$$

Further

$$\left(\Gamma_n^{(i)}\right)^c = \bigcup_{\substack{\ell=1 \\ \ell \neq i}}^{m} \left(\Gamma_n^{(i\ell)}\right)^c , \tag{4.50}$$

and by the definition of the random variable Y_n

$$\omega \in \left(\Gamma_n^{(i\ell)}\right)^c, \ \ell \neq i \Rightarrow |\tilde{Y}_n^{(i\ell)}(\omega)| > \gamma_n , \tag{4.51}$$

which, in turn, implies that either $V_n^{(i)}(\omega) = 1$ or $V_n^{(\ell)}(\omega) = 1$. Hence either

$$|\tilde{Y}_n^{(i\lambda)}(\omega)| > \gamma_n \text{ for all } \lambda = 1,\ldots,m;\ \lambda \neq i ,$$

or

$$|\tilde{Y}_n^{(i\lambda)}(\omega)| = 0 \text{ for all } \lambda = 1,\ldots,m,\ \lambda \neq \ell .$$

Therefore,

$$\omega \in \bigcup_{\substack{\ell=1 \\ \ell \neq i}}^{m} \left(\Gamma_n^{(i\ell)}\right)^c$$

implies that either $|\tilde{Y}_n^{(ij)}(\omega)| > \gamma_n$ or $|\tilde{Y}_n^{(ij)}(\omega)| = 0$ so that

$$|\tilde{Y}_n^{(ij)}| I_{\left(\Gamma_n^{(i)}\right)^c} \le |\tilde{Y}_n^{(ij)}| I_{\left(\Gamma_n^{(ij)}\right)^c} . \tag{4.52}$$

However,

$$E\left\{|\tilde{Y}_n^{(ij)}| I_{\left(\Gamma_n^{(ij)}\right)^c}\right\} \le \gamma_n^{1-r} E|\tilde{Y}_n^{(ij)}|^r \le (4C_1)^r m^{r-1}(\gamma_n p_n)^{1-r}$$

by assumption (4 30) and Lemma 2 (4.15), so that (4.49) becomes

$$\left| E\left\{\tilde{Y}_n^{(ij)} I_{H_n^{(i)}} \middle| \mathcal{Y}_{n-1}\right\} - E\left\{\tilde{Y}_n^{(ij)} I_{H_n^{(i)} \cap \Gamma_n^{(i)}} \middle| \mathcal{Y}_{n-1}\right\}\right|$$

$$\le (4C_1)^r m^{r-1}(\gamma_n p_n)^{1-r} \qquad \text{a.s.} \tag{4.53}$$

Next, let us denote for $\ell = 1,\ldots,m$

$$H_n^{(i\ell)} = \left\{\sum_{k=0}^{n-1} \tilde{Y}_k^{(i\ell)} < -\tilde{Y}_n^{(i\ell)} - \sum_{k=0}^{n} \Delta_k^{(i\ell)}\right\}, \tag{4.54}$$

$$\overline{H}_n^{(i\ell)} = \left\{\sum_{k=0}^{n-1} \tilde{Y}_k^{(i\ell)} < \gamma_n - \sum_{k=0}^{n} \Delta_k^{(i\ell)}\right\}, \tag{4.55}$$

$$\underline{H}_n^{(i\ell)} = \left\{\sum_{k=0}^{n-1} \tilde{Y}_k^{(i\ell)} < -\gamma_n - \sum_{k=0}^{n} \Delta_k^{(i\ell)}\right\}, \tag{4.56}$$

where the numbers $\Delta_k^{(i\ell)}$ are defined by (4.10). Also let

$$\overline{H}_n^{(i)} = \bigcap_{\substack{\ell=1 \\ \ell\ne i}}^{m} \overline{H}_n^{(i\ell)}, \quad \underline{H}_n^{(i)} = \bigcap_{\substack{\ell=1 \\ \ell\ne i}}^{m} \underline{H}_n^{(i\ell)} . \tag{4.57}$$

It is easily seen that

$$H_n^{(i)} = \bigcap_{\substack{\ell=1 \\ \ell\ne i}}^{m} H_n^{(i\ell)} , \tag{4.58}$$

and for $\ell \neq i$

$$\underline{H}_n^{(i\ell)} \subset H_n^{(i\ell)} \cap \Gamma_n^{(i\ell)} \subset \bar{H}_n^{(i\ell)} \quad , \tag{4.59}$$

whence

$$\underline{H}_n^{(i)} \subset H_n^{(i)} \cap \Gamma_n^{(i)} \subset \bar{H}_n^{(i)} \quad . \tag{4.60}$$

Thus

$$\tilde{Y}_n^{(ij)} I_{H_n^{(i)} \cap \Gamma_n^{(i)}} = \tilde{Y}_n^{(ij)+} I_{H_n^{(i)} \cap \Gamma_n^{(i)}} - \tilde{Y}_n^{(ij)-} I_{H_n^{(i)} \cap \Gamma_n^{(i)}}$$

$$\leq \tilde{Y}_n^{(ij)+} I_{\bar{H}_n^{(i)}} - \tilde{Y}_n^{(ij)-} I_{\underline{H}_n^{(i)}} \leq |\tilde{Y}_n^{(ij)}| I_{\bar{H}_n^{(i)} - \underline{H}_n^{(i)}} + \tilde{Y}_n^{(ij)} I_{\underline{H}_n^{(i)}} \quad , \tag{4.61}$$

and, similarly,

$$\tilde{Y}_n^{(ij)} I_{H_n^{(i)} \cap \Gamma_n^{(i)}} \geq - |\tilde{Y}_n^{(ij)}| I_{\bar{H}_n^{(i)} - \underline{H}_n^{(i)}} + \tilde{Y}_n^{(ij)} I_{\underline{H}_n^{(i)}} \quad . \tag{4.62}$$

Since, by definition, both $\bar{H}_n^{(i)}$ and $\underline{H}_n^{(i)}$ are $\mathcal{Y}_{n-1}$-measurable sets we have

$$E\left\{\tilde{Y}_n^{(ij)} I_{\underline{H}_n^{(i)}} \middle| \mathcal{Y}_{n-1}\right\} = I_{\underline{H}_n^{(i)}} E\left\{\tilde{Y}_n^{(ij)}\right\} = 0 \qquad \text{a.s.} \quad , \tag{4.63}$$

and by Lemma 2 (4.15),

$$E\left\{|\tilde{Y}_n^{(ij)}| I_{\bar{H}_n^{(i)} - \underline{H}_n^{(i)}} \middle| \mathcal{Y}_{n-1}\right\}$$

$$= I_{\bar{H}_n^{(i)} - \underline{H}_n^{(i)}} E\left\{|\tilde{Y}_n^{(ij)}|\right\} \leq 4C_1 \, I_{\bar{H}_n^{(i)} - \underline{H}_n^{(i)}} \qquad \text{a.s.} \tag{4.64}$$

We are now going to show that for suitable choice of the truncating sequence $\{\gamma_n\}$

$$I_{\bar{H}_n^{(i)} - \underline{H}_n^{(i)}} \xrightarrow{P} 0 \text{ as } n \to \infty \text{ uniformly in } \underline{\vartheta} \in \Theta \quad , \tag{4.65}$$

which is equivalent to showing that

$$P\left(\overline{H}_n^{(i)} - \underline{H}_n^{(i)}\right) \to 0 \text{ as } n \to \infty \text{ uniformly in } \underline{\vartheta}\epsilon\Theta . \tag{4.66}$$

First, since

$$\overline{H}_n^{(i)} - \underline{H}_n^{(i)} \subset \bigcup_{\substack{\ell=1 \\ \ell\neq i}}^{m} \left(\overline{H}_n^{(i\ell)} - \underline{H}_n^{(i\ell)}\right) , \tag{4.67}$$

$$P\left(\overline{H}_n^{(i)} - \underline{H}_n^{(i)}\right) \leq \sum_{\substack{\ell=1 \\ \ell\neq i}}^{m} P\left(\overline{H}_n^{(i\ell)} - \underline{H}_n^{(i\ell)}\right) . \tag{4.68}$$

Next, the set-theoretical difference $\overline{H}_n^{(i\ell)} - \underline{H}_n^{(i\ell)}$ can be written as

$$\overline{H}_n^{(i\ell)} - \underline{H}_n^{(i\ell)} = \left\{ -(n-1)^{-1/2} \gamma_n - (n-1)^{-1/2} \sum_{k=0}^{n} \Delta_k^{(i\ell)} \leq Z_{n-1}^{(i\ell)} \right.$$

$$\left. < (n-1)^{-1/2} \gamma_n - (n-1)^{-1/2} \sum_{k=0}^{n} \Delta_k^{(i\ell)} \right\} , \ n = 2,3,\ldots , \tag{4.69}$$

and since there is no loss of generality by assuming $p_n \leq m^{-1}$, $n = 0,1,\ldots,$ Lemma 3, together with (4.68), yields

$$P\left(\overline{H}_n^{(i)} - \underline{H}_n^{(i)}\right) \leq \gamma_n (2m)^{1/2} \left(\sum_{k=0}^{n-1} p_k^{-1}\right)^{-1/2}$$

$$+ 32C_1^3 \beta m^{3/2} \sum_{k=0}^{n} p_k^{-2} \left(\sum_{k=0}^{n} p_k^{-1}\right)^{-3/2} \quad ; n = 2,3,\ldots . \tag{4.70}$$

From the conditions (4.31) and (4.32), it follows that the sequence $\{p_n^{-1}: n = 0,1,\ldots\}$ satisfies the hypothesis of Lemma 5. The statement then implies that there is a positive constant $c_o \leq 1$ such that for every $n = 1,2,\ldots$

$$c_o n p_{n-1}^{-1} \le \sum_{k=0}^{n-1} p_k \; . \tag{4.71}$$

Moreover, by (4.31) also

$$\sum_{k=0}^{n-1} p_k^{-\lambda} \le n p_n^{-\lambda}; \; \lambda = 1,2 \; . \tag{4.72}$$

Applying these two inequalities to (4.70), we obtain

$$P\left(\bar{H}_n^{(i)} - \underline{H}_n^{(i)}\right) \le \gamma_n (2m)^{1/2} \, c_o^{-1/2} \, n^{-1/2} \, p_n^{1/2}$$

$$+ \, 32C_1^3 \, c_o^{-3/2} \beta \, m^{3/2} \, (np_n)^{-1/2} \quad ; \; n = 2,3,\ldots \; , \tag{4.73}$$

and trivially also for $n = 1$ (for c_o small enough if necessary).

We will now select the constants γ_n. To minimize both (4.53) and the first term on the right-hand side of (4.73) simultaneously, we require γ_n to satisfy the equation

$$(4C_1)^r \, m^{r-1} (\gamma_n p_n)^{1-r} = 4C_1 \gamma_n (2m)^{1/2} \, c_o^{-1/2} \, n^{-1/2} \, p_n^{1/2} \; . \tag{4.74}$$

For such a γ_n, the inequality (4.73) becomes

$$P\left(\bar{H}_n^{(i)} - \underline{H}_n^{(i)}\right) \le (4C_1)^2 \, 2^{1/2} \, c_o^{-1/2} \, m^{3/2} \, (np_n)^{1/2r - 1/2}$$

$$+ \, 32C_1^3 \, c_o^{-3/2} \beta \, m^{3/2} \, (np_n)^{-1/2} \quad , \; n = 1,2,\ldots \; , \tag{4.75}$$

and the right-hand side of the inequality (4.53) becomes

$$(4C_1)^2 \, 2^{1/2} \, c_o^{-1/2} \, m^{3/2} \, (np_n)^{1/2r - 1/2} \quad , \; n = 1,2,\ldots \; . \tag{4.76}$$

Now it is obvious that both these bounds tend to zero as $n \to +\infty$ so that (4.66) holds, and by (4.53) we conclude that (4.45) goes to zero in probability uniformly in $\underline{\vartheta} \in \Theta$. Finally, it is easily seen that exactly the same reasoning applies if we replace the set $H_n^{(i)}$ and all the sets related to it by the set $K_n^{(i)}$ and similarly related analogs.

Therefore also

$$E\left\{\tilde{Y}_n^{(ij)} I_{K_n^{(i)}} \middle| \mathcal{Y}_{n-1}\right\} \xrightarrow{P} 0 \text{ uniformly in } \underline{\vartheta}\epsilon\Theta^{\infty} , \tag{4.77}$$

which in view of (4.44) yields (4.39).

<u>The proof of Lemma 6 is terminated</u>.

<u>Lemma 7</u>. Let the hypothesis of Lemma 6 be satisfied. Then there is a finite constant C_2 such that for every $n = 1,2,\ldots$ and $\underline{\vartheta}\epsilon\Theta$

$$|E\{\tilde{Y}_n \cdot S_n'\}| \leq C_2 m^{5/2}(np_n)^{1/2r - 1/2} . \tag{4.78}$$

<u>Proof</u>.

Since the random vectors $\tilde{Y}_n$ are centered at expectations, the identity

$$\tilde{Y}_n \cdot S_n' = \frac{1}{m}\sum_{i=1}^{m}\sum_{j=1}^{m} \tilde{Y}_n^{(ij)} S_n'^{(i)} + \frac{1}{m}\sum_{i=1}^{m} \tilde{Y}_n^{(i)} \tag{4.79}$$

yields

$$|E\{\tilde{Y}_n \cdot S_n'\}| \leq \frac{1}{m} \max_{\substack{i=1,\ldots,m \\ j=1,\ldots,m}} |E\{\tilde{Y}_n^{(ij)} S_n'^{(i)}\}| . \tag{4.80}$$

Proceeding similarly as in the proof of Lemma 6, we conclude that both the inequalities (4.44) and (4.53) hold with conditional expectations replaced by unconditional ones. The same is true for the relations (4.63) and (4.64) so that (4.75) and (4.76), together with the remark at the end of the proof of the previous lemma, gives for every $i \neq j$, $n = 1,2,\ldots$, $\underline{\vartheta}\epsilon\Theta^{\infty}$,

$$|E\{\tilde{Y}_n^{(ij)} S_n'^{(i)}\}| \leq C_2 m^{3/2}(np_n)^{1/2r - 1/2} , \tag{4.81}$$

where the constant C_2 includes the constants C_1, c_o, β.

Lemma 7 is proven.

Lemma 8. Let the hypothesis of Lemma 6 be satisfied with the assumption $(\mathfrak{U};r)$ replaced by the assumption $(\mathfrak{U};\infty)$. Then there is a finite constant C_3 such that for every $n = 1,2,\ldots$ and $\underline{\vartheta}\in\Theta$

$$|E(\tilde{Y}_n \cdot S_n')| \leq C_3 m^{5/2}(np_n)^{-1/2} . \tag{4.82}$$

Proof.

The proof is only a modification of the proofs of Lemma 7 and Lemma 6. Clearly, $(\mathfrak{U};\infty)$ implies $(\mathfrak{U};3)$ so that the hypotheses of the two previous lemmas are satisfied. Moreover, under $(\mathfrak{U};\infty)$ there is a finite constant C_4 such that for all $i = 1,\ldots,m$; $n = 0,1,\ldots$;

$$|\tilde{Y}_n^{(i)}| \leq C_4 m p_n^{-1} \qquad \text{a.s.} \tag{4.83}$$

and consequently also for all $j = 1,\ldots,m$

$$|\tilde{Y}_n^{(ij)}| \leq 2C_4 m p_n^{-1} \qquad \text{a.s.} \tag{4.84}$$

Thus we can set

$$\gamma_n = 2C_4 m p_n^{-1} , \quad n = 1,2,\ldots$$

in (4.46) whence in (4.48)

$$I_{\left(\Gamma_n^{(i)}\right)^c} = 0 \qquad \text{a.s.} \tag{4.85}$$

so that the right-hand side of (4.49) is zero and the first term on the right-hand side of (4.73) becomes

$$2^{3/2}\, C_4\, c_o^{-1/2}\, m^{3/2}\, (np_n)^{-1/2} . \tag{4.86}$$

This implies

$$|E(\tilde{Y}_n^{(ij)} S_n'^{(i)})| \leq C_3 m^{3/2} (np_n)^{-1/2} \tag{4.87}$$

for every $i \neq j$; $n = 1,2,\ldots$; $\underline{\vartheta} \in \Theta^{\infty}$, which together with (4.80) proves Lemma 8.

Chapter V

MAIN THEOREMS

<u>Theorem 1</u>. Let $\{\Psi_n: n = 1,2,\ldots\}$ be the sequence of pure strategies generated by the strategic rule (*) defined in Chapter III; let the sequence of probabilities (3.2) satisfy the conditions

$$p_n \downarrow 0 , \tag{5.1}$$

and

$$np_n \uparrow +\infty \text{ as } n \to +\infty ; \tag{5.2}$$

let the assumptions (2.1), and $(\mathcal{W};r)$ for some $r \geq 3$ be satisfied. Then

$$\frac{1}{n} \sum_{k=1}^{n} W_k(\vartheta_k,\Psi_k) - \varphi(\tau_n) \xrightarrow{P} 0 \tag{5.3}$$

uniformly in all sequences $\underline{\vartheta}\in\Theta^{\infty}$.

<u>Proof</u>.

Let $\underline{\vartheta}$ be an arbitrary sequence from Θ^{∞}; let for $n = 1,2,\ldots, \mathcal{F}_n$ be the σ-field induced by the family of random vectors

$$\{W_o(\vartheta_o,.),\ldots,W_n(\vartheta_n,.);\ S_1,\ldots,S_{n+1}\} , \tag{5.4}$$

where S_n is defined by (*).

By (2.16) and (2.17) we have for every $n = 1,2,\ldots$

$$E\{W_n(\vartheta_n,\Psi_n)|\mathcal{F}_{n-1}\} = w(\vartheta_n,S_n) \qquad \text{a.s.} \tag{5.5}$$

Furthermore, $(\mathcal{W};r)$, $r \geq 3$, implies that for every $n = 1,2,\ldots, \underline{\vartheta}\in\Theta^{\infty}$ and $1 \leq r' \leq r$

$$E|W_n(\vartheta_n,S_n)|^{r'} \leq C_o^{r'} . \tag{5.6}$$

Therefore, by Lemma 4, (5.3) is equivalent to

$$\frac{1}{n}\sum_{k=1}^{n} w(\vartheta_n, S_n) - \varphi(\tau_n) \xrightarrow{P} 0 \quad \text{uniformly in } \underline{\vartheta}\in\Theta^{\infty} . \tag{5.7}$$

Let $\{S_n': n = 0,1,\ldots\}$ be a sequence of random vectors defined by

$$S_n' = s^*\left(\sum_{k=0}^{n} Y_k\right) , \tag{5.8}$$

where $\{Y_n: n = 0,1,\ldots\}$ is the sequence (3.5), and let us denote

$$\left.\begin{aligned} X_n &= \frac{1}{n}\sum_{k=1}^{n} w(\vartheta_n, S_n) - \varphi(\tau_n) \\ X_n' &= \frac{1}{n}\sum_{k=1}^{n} w(\vartheta_n, S_n') - \varphi(\tau_n) \\ \text{and} \quad X_n'' &= \frac{1}{n}\sum_{k=1}^{n} w(\vartheta_n, S_{n-1}') - \varphi(\tau_n) \end{aligned}\right\} . \tag{5.9}$$

We have

$$|X_n - X_n''| \le \frac{1}{n}\sum_{k=1}^{n} |w(\vartheta_k, S_k) - w(\vartheta_k, S_{k-1}')| \tag{5.10}$$

and

$$|X_n - X_n'| \le \frac{1}{n}\sum_{k=1}^{n} |w(\vartheta_k, S_k) - w(\vartheta_k, S_k')| . \tag{5.11}$$

Let $\{U_n: n = 0,1,\ldots\}$ be the sequence of random variables (3.1). Since by (*) and (5.8), $U_n = 0 \Rightarrow S_n = S_{n-1}'$, and by (3.6) and (5.8), $U_n = 0 \Rightarrow Y_n = 0 \Rightarrow S_n' = S_{n-1}'$, we conclude that

$$|w(\vartheta_n, S_n) - w(\vartheta_n, S_{n-1}')| > 0 \Rightarrow U_n = 1 , \tag{5.12}$$

and also

$$|w(\vartheta_n, s_n) - w(\vartheta_n, s_n')| > 0 \Rightarrow U_n = 1 \ . \tag{5.13}$$

From this and (5.6), it follows that the right-hand sides of both (5.10) and (5.11) are both bounded by

$$2C_o \frac{1}{n} \sum_{k=1}^{n} U_k \ ,$$

which goes to zero almost surely because of the condition (5.1). Thus we have

$$|X_n - X_n'| \overset{a.s.}{\rightarrow} 0 \quad \text{and} \quad |X_n - X_n''| \overset{a.s.}{\rightarrow} 0 \ , \tag{5.14}$$

both uniformly in $\underline{\vartheta} \in \Theta^{\infty}$.

We are going to prove that the random variables X_n' are bounded from above and the random variables X_n'' from below by random variables that both tend to zero in probability uniformly in $\underline{\vartheta} \in \Theta^{\infty}$. This, in view of (5.14), will prove (5.7).

Let us start with X_n'. Using (2.14), we obtain

$$\begin{aligned}
X_n' &= \frac{1}{n} \sum_{k=1}^{n} [kw(\tau_k, s_k') - (k-1)w(\tau_k, s_k')] - \varphi(\tau_n) \\
&= \frac{1}{n} \left[\sum_{k=1}^{n} kw(\tau_k, s_k') - \sum_{k=1}^{n} kw(\tau_k, s_{k+1}') - n\varphi(\tau_n) \right] \\
&= \frac{1}{n} \sum_{k=1}^{n-1} k[w(\tau_k, s_k') - w(\tau_k, s_{k+1}')] + n[w(\tau_n, s_n') - \varphi(\tau_n)] \ ,
\end{aligned}$$

$$n = 1,2,\ldots \ . \tag{5.15}$$

Let

$$Z_n = n^{-1/2} \sum_{k=0}^{n} \tilde{Y}_k ; \ n = 1,2,\ldots ; \ Z_o = 0 \ , \tag{5.16}$$

where $\tilde{Y}_n = Y_n - E\{Y_n\}$, $n = 0,1,\ldots$. By the definition of the mapping s^* and by (4.1), we have

$$S'_n = s^*\left(\frac{1}{n}\sum_{k=0}^{n} E\{Y_k\} + \frac{1}{n}\sum_{k=0}^{n} \tilde{Y}_k\right) = s^*\left(w(\tau_n,.) + n^{-1/2} Z_n + \eta\,\frac{n+1}{n}\right)$$

$$= s^*\left(w(\tau_n,.) + n^{-1/2} Z_n\right), \quad n = 1,2,\ldots , \tag{5.17}$$

where $\eta \in R^m$ is the vector with all components equal to one. Applying now Lemma 1 with $x = k^{-1/2} Z_k$, $\alpha_1 = S'_k$, $\alpha_2 = S'_{k+1}$ to the summands in (5.15) and with $x = n^{-1/2} Z_n$, $\alpha_1 = s^*\big(w(\tau_n,.)\big)$, $\alpha_2 = S'_n$ we obtain in view of (5.17) and (2.6) the relation

$$X'_n \le \frac{1}{n}\sum_{k=1}^{n-1} k^{-1/2} Z_k\cdot(S'_{k+1} - S'_k) + n^{-1/2} Z_n\cdot\left(s^*\big(w(\tau_n,.)\big) - S'_n\right)$$

$$= \frac{1}{n}\sum_{k=1}^{n}\left((k-1)^{1/2} Z_{k-1} - k^{1/2} Z_k\right)\cdot S'_k + n^{-1/2} Z_n\cdot s^*\big(w(\tau_n,.)\big) . \tag{5.18}$$

However by (5.16)

$$(k-1)^{1/2} Z_{k-1} - k^{1/2} Z_k = -\tilde{Y}_k \;; \quad k = 1,2,\ldots \;; \tag{5.19}$$

so that (5.18) becomes

$$X'_n \le -\frac{1}{n}\sum_{k=1}^{n} \tilde{Y}_k\cdot S'_k + n^{-1/2} Z_n\cdot s^*\big(w(\tau_n,.)\big) . \tag{5.20}$$

The first term on the right-hand side of (5.20) goes to zero in probability uniformly in $\underset{\sim}{\vartheta}\in\Theta^\infty$ by Lemma 6. Next, since $s^*(w(\tau_n,.))\in\mathcal{Q}$,

$$n^{-1/2} Z_n\cdot s^*\big(w(\tau_n,.)\big) \le \max_{i=1,\ldots,m}\left|\frac{1}{n}\sum_{k=0}^{n}\tilde{Y}_k^{(i)}\right| . \tag{5.21}$$

However, the random variables $\tilde{Y}_k^{(i)}$; $k = 0,1,\ldots$; are independent and centered at expectation, and by Lemma 2 (4.14) and (5.1)

$$n^{-2} \sum_{k=0}^{n} |\tilde{Y}_k^{(i)}|^2 \leq (2C_1)^2 \, mn^{-2} \sum_{k=0}^{n} p_k^{-1} \leq (2C_1)^2 \, mn^{-2}(n+1)p_n^{-1} , \quad (5.22)$$

which goes to zero by the condition (5.2). Hence by the weak law of large numbers[†] ([5], p. 234, prop. b)

$$\left| \frac{1}{n} \sum_{k=0}^{n} \tilde{Y}_k^{(i)} \right| \xrightarrow{P} 0 \quad \text{uniformly in } \underset{\sim}{\vartheta} \in \Theta^{\infty} \quad (5.23)$$

for all $i = 1,\ldots,m$. Because of (5.21), the same is true for the last term of (5.20) so that

$$\limsup_{n\to+\infty} X_n' = 0 \quad \text{in probability uniformly in } \underset{\sim}{\vartheta} \in \Theta^{\infty} . \quad (5.24)$$

It remains to show that also

$$\liminf_{n\to+\infty} X_n'' = 0 \quad \text{in probability uniformly in } \underset{\sim}{\vartheta} \in \Theta^{\infty} . \quad (5.25)$$

For this, let us write using again (2.6)

$$X_n'' = \frac{1}{n} \sum_{k=1}^{n} k[w(\tau_k, S_{k-1}') - w(\tau_k, S_k')] + w(\tau_n, S_n') - \varphi(\tau_n) . \quad (5.26)$$

Using (5.17) and applying again Lemma 1 to the summands, this time with $\alpha_2 = S_{k-1}'$, we obtain

$$w(\tau_k, S_{k-1}') - w(\tau_k, S_k') \geq k^{1/2} \, z_k \cdot (S_k' - S_{k-1}') . \quad (5.27)$$

[†] Or Lemma 4 with $\mathcal{F}_n$ induced by $\{\tilde{Y}_o^{(i)},\ldots,\tilde{Y}_n^{(i)}\}$.

Further by (2.6)

$$w(\tau_n, S'_n) - \varphi(\tau_n) \geq 0 \;, \tag{5.28}$$

so that

$$X''_n \geq \frac{1}{n} \sum_{k=1}^{n} k^{1/2} \, Z_k \cdot (S'_k - S'_{k-1})$$

$$= \frac{1}{n} \sum_{k=1}^{n} \left((k-1)^{1/2} \, Z_{k-1} - k^{1/2} \, Z_k \right) \cdot S_{k-1} + n^{-1/2} \, Z_n \cdot S'_n \;. \tag{5.29}$$

Hence by (5.19) and the fact that $S'_n \in \mathcal{U}$, we obtain

$$X''_n \geq - \frac{1}{n} \sum_{k=1}^{n} \tilde{Y}_k \cdot S'_{k-1} - \max_{i=1,\ldots,m} \left| \frac{1}{n} \sum_{k=0}^{n} \tilde{Y}_k^{(i)} \right| \; ; \quad n = 1,2,\ldots \;. \tag{5.30}$$

Let, for every $n = 0,1,\ldots$, $\mathcal{Y}_n$ denote the σ-field induced by the family $\{Y_o,\ldots,Y_n\}$. Since the random vectors $\tilde{Y}_n$ and S'_{n-1} are independent and S'_n is $\mathcal{Y}_n$-measurable, we have for all $n = 1,2,\ldots$

$$E\{\tilde{Y}_n \cdot S'_{n-1} | \mathcal{Y}_{n-1}\} = E\{\tilde{Y}_n\} \cdot S'_{n-1} = 0 \qquad \text{a.s.} \tag{5.31}$$

Hence, by Lemma 4 and (5.22),

$$\frac{1}{n} \sum_{k=1}^{n} \tilde{Y}_k \; S'_{k-1} \xrightarrow{P} 0 \quad \text{uniformly in } \mathfrak{z} \in \Theta \;,$$

and by (5.23) the same is true for the last term in (5.30). Thus (5.25) holds and <u>Theorem 1 is proven</u>.

Because of assumption $(\mathcal{W};r)$, the dominated convergence theorem implies that under the conditions of Theorem 1 also

$$E\left\{\frac{1}{n}\sum_{k=1}^{n} W_k(\vartheta_k,\Psi_k)\right\} - \varphi(\tau_n) \to 0 \quad \text{uniformly in } \underline{\vartheta}\epsilon\Theta . \tag{5.32}$$

It is, nevertheless, of interest to investigate the rate of this convergence.

Theorem 2. Let the hypothesis of Theorem 1 be satisfied.

Then

$$\left|E\left\{\frac{1}{n}\sum_{k=1}^{n} W_k(\vartheta_k,\Psi_k)\right\} - \varphi(\tau_n)\right| = 0\left(\max\left\{(\overline{p_n}),\ \left(\overline{(np_n)^{-1/2+1/2r}}\right)\right\}\right) \tag{5.33}$$

uniformly in all sequences $\underline{\vartheta}\epsilon\Theta^{\infty}$.

Proof.

Proceeding exactly as in the proof of Theorem 1, we observe that the random variables (5.9) satisfy for every n = 1,2,... the inequality

$$X_n'' - 2C_o \frac{1}{n}\sum_{k=1}^{n} U_k \le X_n \le X_n' + 2C_o \frac{1}{n}\sum_{k=1}^{n} U_k . \tag{5.34}$$

Since the random variables U_n take values 0 and 1 with $P\{U_n = 1\} = p_n$, we have

$$E\{X_n''\} - 2C_o \frac{1}{n}\sum_{k=1}^{n} p_k \le E\{X_n\} \le E\{X_n'\} + 2C_o \frac{1}{n}\sum_{k=1}^{n} p_k . \tag{5.35}$$

Furthermore, the inequality (5.20), Lemma 7, and the fact that $E\{Z_n\} = 0$ imply

$$E\{X_n'\} \le C_2 m^{5/2} \frac{1}{n}\sum_{k=1}^{n} (kp_k)^{-1/2+1/2r} , \tag{5.36}$$

and the inequality (5.30), independence of $\tilde{Y}_k$ and S_{k-1}', and $E\{\tilde{Y}_k\} = 0$, $k = 1,2,\ldots$, give

$$E\{X''_n\} \geq - \max_{i=1,\ldots,m} E\left|\frac{1}{n}\sum_{k=1}^{n} \tilde{Y}_k^{(i)}\right| . \tag{5.37}$$

Next, by Jensen Inequality, Bienaymé Equality, and (5.22) we have for every $i = 1,\ldots,m$

$$\left(E\left|\frac{1}{n}\sum_{k=1}^{n} \tilde{Y}_k^{(i)}\right|\right)^2 \leq E\left|\frac{1}{n}\sum_{k=1}^{n} \tilde{Y}_k^{(i)}\right|^2 = \frac{1}{n}\sum_{k=1}^{n} E\left|\tilde{Y}_k^{(i)}\right|^2$$

$$\leq (2C_1)^2\, m(np_n)^{-1} , \tag{5.38}$$

so that (5.35) becomes

$$- 2C_1 m^{1/2}\, (np_n)^{-1/2} - 2C_o\, \frac{1}{n}\sum_{k=1}^{n} p_k \leq E\{X_n\}$$

$$\leq C_2 m^{5/2}\, \frac{1}{n}\sum_{k=1}^{n} (kp_k)^{-1/2 + 1/2r} + 2C_o\, \frac{1}{n}\sum_{k=1}^{n} p_k . \tag{5.39}$$

However, by (5.1), (5.2), and $r \geq 3$,

$$\frac{1}{n}\sum_{k=1}^{n} (kp_k)^{-1/2 + 1/2r} \geq (np_n)^{-1/2} \tag{5.40}$$

so that the first term in (5.39) is lower-bounded by

$$- \frac{1}{n}\sum_{k=1}^{n} (kp_k)^{-1/2 + 1/2r} .$$

<u>Theorem 2 is proven</u>.

Theorem 3. Let the hypothesis of Theorem 1 be satisfied with the assumption $(\mathcal{U};r)$ replaced by the assumption $(\mathcal{U};\infty)$. Then

$$\left|E\left\{\frac{1}{n}\sum_{k=1}^{n} W_k(\vartheta_k,\Psi_k)\right\} - \varphi(\tau_n)\right| = O\left(\max\left\{(\overline{p_n}),\ \left(\overline{(np_n)^{-1/2}}\right)\right\}\right) \tag{5.41}$$

uniformly in all sequences $\underline{\vartheta}\epsilon\Theta$.

Proof.

The proof is verbatim that of Theorem 2, except that Lemma 8 is used instead of Lemma 7 so that the exponent 1/2r vanishes from (5.36), (5.39), and (5.40), and the constant C_2 is replaced by the constant C_3 of Lemma 8.

To the end of this chapter, let us consider the case in which the sequences $\underline{\vartheta}$ of Nature's strategies are sequences of independent identically distributed random variables. More precisely, let us suppose that Nature at the beginning of the sequence of plays selects a probability measure τ from a class T_1 of probability measures defined on the measurable space $(\Theta,\mathcal{T}_1)$. The class T_1 and the σ-field $\mathcal{T}_1$, not necessarily identical with those introduced in Chapter II, are supposed to be given, however, unknown to the player. We also assume that the random loss function $\mathcal{U}$ is such that for every $a\epsilon A$, $\tau\epsilon T_1$, $W(.,a)$ is for almost every $\omega\epsilon\Omega$ a $\mathcal{T}_1$-measurable and τ-integrable function.

In this setup, let $\tau\epsilon T_1$ and let

$$\{\Phi_n:\ n = 1,2,\ldots\} \tag{5.42}$$

be a sequence of independent identically distributed random mappings from $(\Omega,\mathcal{A})$ into $(\Theta,\mathcal{T}_1)$ such that $P\Phi_n^{-1} = \tau$ and such that the sequences (5.40), (3.1), (3.3), and (2.15) for any fixed $\underline{\vartheta}\epsilon\Theta^{\infty}$ are mutually independent.

Theorem 1 then yields immediately

$$\frac{1}{n}\sum_{k=1}^{n} W_k(\Phi_k,\Psi_k) - \varphi(\tau) \xrightarrow{P} 0 \quad \text{uniformly in } \tau\epsilon T_1 . \tag{5.43}$$

However, we may establish a stronger convergence in this case.

<u>Theorem 4</u>. Let $\{\Psi_n: n = 1,2,\ldots\}$ be a sequence of pure strategies generated by the strategic rule (*) of Chapter III; let the sequence of probabilities (3.2) satisfy the conditions

$$p_n \downarrow 0 \tag{5.44}$$

and

$$n^{1-\epsilon} p_n \uparrow +\infty \quad \text{for some } \epsilon > 0 ; \tag{5.45}$$

let the assumptions (2.2) and ($\mathcal{W}$;2) be satisfied. Then

$$\frac{1}{n}\sum_{k=1}^{n} W_k(\Phi_k,\Psi_k) - \varphi(\tau) \xrightarrow{a.s.} 0 \tag{5.46}$$

uniformly in all $\tau\epsilon T_1$.

<u>Proof</u>.

Let $\tau\epsilon T_1$ and let again $\mathcal{F}_n$ denote the σ-field induced by the family (5.4), where now $\Phi_1,\Phi_2,\ldots$ is the sequence (5,42). By the measurability and integrability assumption we made about the random loss function, we have now instead of (5.5)

$$E\{W_n(\Phi_n,\Psi_n)|\mathcal{F}_{n-1}\} = w(\tau,s_n) \qquad \text{a.s.} \tag{5.47}$$

Hence, using the assumption ($\mathcal{W}$;2) and Stability Theorem ([5], p. 387), we conclude that (5.44) is equivalent to showing

$$\frac{1}{n}\sum_{k=1}^{n} w(\tau,s_n) - \varphi(\tau) \xrightarrow{a.s.} 0 \quad \text{uniformly in } \tau\epsilon T_1 . \tag{5.48}$$

Let X_n, X'_n and X''_n be defined as in (5.9) with ϑ_n and τ_n replaced by τ. By the same reasoning as in the proof of Theorem 1, we conclude that (5.14) holds again, now uniformly in $\tau \epsilon T_1$.

By the relation (5.17) and Lemma 1 we have

$$w(\tau, S'_n) - \varphi(\tau) \leq \frac{1}{n} \sum_{k=0}^{n} \tilde{Y}_k \cdot \left[s^*\left(w(\tau,.)\right) - S'_n\right]$$

$$\leq \max_{i=1,\ldots,m} \left| \frac{1}{n} \sum_{k=0}^{n} \tilde{Y}_k^{(i)} \right| , \tag{5.49}$$

and by the assumption $(\mathfrak{W};2)$ and Lemma 2 (4.14) for every $i = 1,\ldots,m$

$$\sum_{k=1}^{n} k^{-2} E|\tilde{Y}_k^{(i)}|^2 \leq (2C_1)^2 m \sum_{k=1}^{n} k^{-2} p_k^{-1} . \tag{5.50}$$

However, by (5.45), the series

$$\sum_{k=1}^{\infty} k^{-2} p_k^{-1}$$

converges so that by the strong law of large numbers ([5], p. 238, prop A) the right-hand side of (5.49) converges to zero almost surely uniformly in $\tau \epsilon T_1$.

As for X''_n, by (5.31), (5.50), and Stability Theorem

$$\frac{1}{n} \sum_{k=1}^{n} \tilde{Y}_k S'_{k-1} \overset{a.s.}{\to} 0 \text{ uniformly in } \tau \epsilon T_1 , \tag{5.51}$$

so that the same is true for both terms on the right-hand side of (5.30).

Therefore, (5.24) and (5.25) hold almost surely uniformly in $\tau \epsilon T_1$, which together with (5.14) terminates the proof.

Theorems 2 and 3 appear also in a stronger version.

Theorem 5. Let $\{\Psi_n: n = 1,2,\ldots\}$ be a sequence of pure strategies generated by the strategic rule (*) of Chapter III; let the sequence of probabilities (3.2) satisfy the conditions (5.1) and (5.2) of Theorem 1; let the assumptions (2.2) and ($\mathcal{W}$;2) hold. Then

$$|E\{W_n(\Phi_n,\Psi_n)\} - \varphi(\tau)| = O(\max\{p_n, (np_n)^{-1/2}\}) \tag{5.52}$$

uniformly in all $\tau \in T_1$.

Proof.

Proceeding as in the proof of Theorem 4, we obtain the inequality

$$E\{w(\tau,S'_{n-1})\} - \varphi(\tau) - 2C_o\, p_n \leq E\{w(\Phi_n,\Psi_n)\} - \varphi(\tau)$$

$$\leq E\{w(\tau,S'_n)\} - \varphi(\tau) + 2C_o\, p_n \; . \tag{5.53}$$

From (5.49) and (5.38), we have

$$E\{w(\tau,S_n)\} - \varphi(\tau) \leq 2C_1\, mn^{-1}\, (n+1)^{1/2}\, p_n^{-1/2} \; . \tag{5.54}$$

Next, by (5.28) and Lemma 1

$$E\{w(\tau,S'_{n-1})\} - \varphi(\tau) \geq E\{w(\tau,S'_{k-1}) - w(\tau,S'_k)\}$$

$$\geq - \max_{i=1,\ldots,m} E\left|\frac{1}{n}\sum_{k=0}^{n} \tilde{Y}_k^{(i)}\right| \tag{5.55}$$

so that by (5.22) also

$$E\{w(\tau,S'_{n-1})\} - \varphi(\tau) \geq -2C_1\, mn^{-1}\, (n+1)^{1/2}\, p_n^{-1/2} \; . \tag{5.56}$$

Hence (5.56), (5.54) and (5.53) give the statement.

The theorem is proven.

Chapter VI

CONCLUDING REMARKS

In this last chapter, let us make a few remarks concerning various assumptions we have made, possible generalizations, and relationships to other problems.

First, let us consider the rate of convergence established in Theorems 2, 3, and 5. It is easy to see that, for example, the choice $p_n = n^{-q}$, $0 < q < 1$, which satisfies both the conditions (5.1) and (5.2) and (5.44) and (5.45), yields the maximum rate $O(n^{-1 - 4r/1-3r})$ under the assumptions of Theorem 2, that is, $O(n^{-1/4})$ in the most unfavorable case of $r = 3$, and the maximum rate $O(n^{-1/3})$ under the assumptions of Theorems 3 and 5. This is, of course, considerably slower than the typical rate $O(n^{-1/2})$ of Blackwell's and Hannan's rules and others derived from them. Notice, however, that the strategic rule suggested in this paper does not make use of all the information available to the player since the information obtained during active plays is disregarded, principally for the sake of simplifying the proofs. It is, nevertheless, conceivable that if the disregarded information were used, the rate of convergence might be improved. One way of doing this may be to record the loss during active plays as well and switch to another strategy as soon as the accumulated loss for the strategy being used decreases under the next largest value of accumulated losses recorded before.

As far as the assumptions of the theorems are concerned, the assumption $(\mathcal{U};r)$ was necessary for the proofs and can hardly be removed unless the method of proofs is changed considerably. The assumption (2.2) is merely a technical matter, as mentioned earlier. The remaining assumption (2.1) was introduced to keep the variance of the random variables $\tilde{Y}_n^{(i)}$ nonzero (see Lemmas 2 and 3). For the same reason, the vector η [see (3.6)] appeared in the definition of the strategic rule (*). Some considerations indicate, however, that the same effect would be achieved without the assumption (2.1) if we replaced the vector η in (3.6) by the random vector with i-th component $(i = 1,\ldots,m)$ equal to sign $(W_n(\vartheta_n, a^{(i)}))$. Notice also that neither the assumption (2.1) nor the vector η is needed to establish the truth of Theorems 4 and 5.

In Theorem 1, we have proven uniform convergence in probability. Naturally, a question arises of whether a stronger convergence--i.e., whether convergence almost surely can be proven. This may quite be the case; we were, however, unable to do this here and the question remains open.

Next, let us discuss briefly the possibility of generalizing the results to infinite games (i.e., when the set A is infinite). For A denumerable, this might in principle be done by letting the random vectors V_n take values in finite subsets $A_n \subset A$ (again with uniform distribution on the vertices of the simplex spanned on A_n) and then let m_n, the cardinality of A_n, tend to infinity slowly enough. It is clear, however, that in this case the convergence in Theorems 1-5 may not be uniform unless the generic game possesses some properties (e.g., to be totally bounded in the sense of Wald's metric) that makes it approximable by finite games. This is even more evident for the case in which A is uncountable. Nevertheless, there is a large class of games with the above properties (e.g., games on the unit square, polynomial games, etc.) so that some investigation in this direction might be worthwhile.

We would also like to mention that the problem studied here is closely related to the so-called "two-armed bandit problem" (see, e.g., ref. [7]); in fact, it includes the latter as a degenerate case. The two-armed (or more generally m-armed) bandit problem can be briefly described as follows: Given are m independent random experiments with outcomes 0 (success) and 1 (failure) having probabilities $1 - \pi_i$ and π_i; $i = 1,\ldots,m$; respectively, which are either unknown or to which of the m experiments a particular pair $(\pi_i, 1 - \pi_i)$ belongs is unknown. These experiments are independently repeated; at each step only one of them is allowed to be performed. The problem is to find a rule for performing these experiments that would minimize the expected average number of failures. Clearly, no rule can do better than to achieve $\min\{\pi_1,\ldots,\pi_m\}$ which is nothing but the value of the minimum functional φ of the game with random loss function (Θ, A, W), where Θ is one-element set $\{\theta\}$ and

$$W(\theta, a^{(i)}) = \begin{cases} 1 & \text{with probability } \pi_i \\ 0 & \text{with probability } 1-\pi_i \end{cases} \qquad i = 1,\ldots,m \ .$$

Thus our rule can be used for this problem and we conclude from Theorem 4 that the average number of failures converges almost surely to $\min\{\pi_1, \ldots, \pi_m\}$ while Theorem 5 states that the expectation of a failure in n-th experiment converges to $\min\{\pi_1, \ldots, \pi_m\}$ with the rate $O(\max\{p_n, (np_n)^{-1/2}\})$.

Finally, let us make a few remarks about the case when, instead of a sequential game against Nature, we consider a sequential game against an opponent (malicious or not); that is, if we allow the strategies δ_n of the first player to depend on the past strategies of the second player and on the losses incurred. This is the case studied recently by A. Baños [1] under the same assumptions about the information available to the second player as we have made and under nearly the same assumptions about the generic game, viz., A finite and $(\mathcal{W};2)$. He succeeded in exhibiting a strategic rule for the player with the property that the average loss is, with probability one, asymptotically not greater than the value of the generic game (= maximum of the minimum functional). However, as can easily be seen, his strategic rule need not have the optimum property (5.3) even in the sequential game against Nature. On the other hand, our rule fails to have this optimum property in a sequential game against an opponent and, in general, does not even guarantee that the average loss will achieve the value of the game. This can be seen from the following simple example (due to T. M. Cover): Consider the generic game of "matching pennies," i.e., $A = \Theta = \{0,1\}$ and $W(\vartheta, a) = w(\vartheta, a) = |\vartheta - a|$ nonrandom, and suppose that the opponent decided to play for each $n = 1,2,\ldots$, $\vartheta_{n+1} = 1$ if $\Psi_n = 0$, and vice versa. Since, according to our rule, the player does not change his strategy between successive active plays and since the condition $p_n \downarrow 0$ implies that long runs of active plays will occur more and more often, the average loss incurred by the player will tend to 1 while the value of the game is 1/2. Thus the only rule known at this time which retains the optimum property (5.3) (in the a.s. sense) in both the sequential game against Nature and against an opponent is the rule of D. Blackwell [3]. The question remains open whether a similar rule exists even when the player's information is so severely limited as assumed in this paper.

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13. ABSTRACT

A repetitive play of a game against Nature is considered under the assumption that the player knows nothing about the game except his own set of strategies. After each play, he is told the value of the random loss incurred by him. A strategic rule for the player is defined with the property that the average loss achieves asymptotically the minimum functional of the game in probability and uniformly in all sequences of Nature's strategies. The rate of convergence of expected average losses is shown as well.

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